Nearly Decomposable Matrices*

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1. INTRODUCTION

The concept of indecomposability is fundamental in the theory of nonnegative matrices (i.e., matrices whose entries are nonnegative). An n-square nonnegative matrix is said to be *partly decomposable* if it contains an $s \times (n - s)$ zero submatrix; otherwise it is *totally indecomposable.* A totally indecomposable matrix $A = (a_{ij})$ is called *nearly decomposable* if, for every positive entry a_{hk}, the matrix $A - a_{hk}E_{hk}$ is partly decomposable (here E_{hk} denotes the n-square matrix with 1 in the (h, k) position and zeros elsewhere).

Nearly decomposable matrices were introduced in [5] by Sinkhorn and Knopp, who proved the fundamental theorem on the structure of nearly decomposable matrices (see Lemma 2). Sinkhorn used it later in [4] to resolve affirmatively a conjecture of M. Hall. In [3] I improved and generalized Sinkhorn's theorem. In this paper I improve somewhat the Sinkhorn-Knopp result (Theorem 1) and obtain an upper bound for the number of positive entries in a nearly decomposable matrix (Theorem 2). This bound is attained only for an interesting special type of matrices that had already made their appearance in a celebrated theorem by de Bruijn and Erdös [1].

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2. PRELIMINARIES

The *permanent* of an n-square matrix is defined by

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Some basic properties of permanents and an extensive bibliography on the theory of permanents can be found in [2]. The submatrix obtained from A by deleting the ith row and jth column of A is denoted by $A(i|j)$. We now quote in a convenient form a well-known theorem due to Frobenius and König. This is followed by a statement of the theorem by Sinkhorn and Knopp.

LEMMA 1 (Frobenius-König). *The permanent of an n-square nonnegative matrix A is zero if and only if A contains an $s \times t$ zero submatrix such that $s + t = n + 1$.*

LEMMA 2 (Sinkhorn-Knopp). *If A is a nearly decomposable nonnegative matrix, then there exist permutation matrices P and Q and an integer $r > 1$ such that*

$$PAQ = \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 & 0 & E_1 \\ E_2 & A_2 & 0 & \cdot & 0 & 0 & 0 \\ 0 & E_3 & A_3 & \cdot & 0 & 0 & 0 \\ & & & \ddots & & \vdots & \\ 0 & 0 & 0 & \cdot & E_{r-1} & A_{r-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & E_r & A_r \end{bmatrix}, \tag{1}$$

where each A_i is nearly decomposable and each E_i has exactly one positive entry.

The converse of Lemma 2 is only partly true: a matrix of the form (1), where the A_i are nearly decomposable and each E_i contains exactly one positive entry, is certainly totally indecomposable [see 5] but not necessarily nearly decomposable.

LEMMA 3. (i) *An n-square nonnegative matrix A is totally indecomposable if and only if*

$$\operatorname{per}(A(i|j)) > 0$$

for $i, j = 1, \ldots, n$.

(ii) *If A is a totally indecomposable* (0, 1) *matrix with row sums* r_i, $i = 1, \ldots, n$, *and column sums* c_j, $j = 1, \ldots, n$, *then*

$$\operatorname{per}(A) \geqslant \max(r_1, \ldots, r_n, c_1, \ldots, c_n).$$

Proof. (i) Since A is nonnegative, $\operatorname{per}(A(i|j)) > 0$ or $\operatorname{per}(A(i|j)) = 0$. By Lemma 1, $\operatorname{per}(A(i|j)) = 0$ if and only if $A(i|j)$ contains an $s \times t$ zero submatrix with $s + t = (n - 1) + 1 = n$. But then A would contain an $s \times (n - s)$ zero submatrix; this is impossible, since A is totally indecomposable.

(ii) Expand $\operatorname{per}(A)$ by the ith row:

$$\operatorname{per}(A) = \sum_{j=1}^{n} a_{ij} \operatorname{per}(A(i|j)).$$

Now, by part (i), $\operatorname{per}(A(i|j)) > 0$. Since A is a (0, 1) matrix, we actually have $\operatorname{per}(A(i|j)) \geqslant 1$. Therefore

$$\begin{aligned} \operatorname{per}(A) &\geqslant \sum_{j=1}^{n} a_{ij} \\ &= r_i. \end{aligned}$$

Similarly we can show that $\operatorname{per}(A) \geqslant c_j$.

Two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to have the *same zero pattern* if $a_{ij} = 0$ whenever $b_{ij} = 0$, and vice versa. Lemmas 1, 2, 3(i), and Theorem 1 are stated in terms of nonnegative matrices. This generality is more apparent than real, however, since any result on the distribution of positive entries in a totally indecomposable zero matrix is no more general than the same theorem specialized to (0, 1) matrices.

3. RESULTS

Our first theorem gives further information on the structure of nearly decomposable matrices.

THEOREM 1. *If A is a nearly decomposable n-square matrix $n \geqslant 3$, and P, Q are permutation matrices such that PAQ is in the form* (1), *then no A_i can be 2-square.*

Proof. Let $PAQ = C = (c_{ij})$ be in the form (1). Suppose that A_i is a 2×2 nearly decomposable matrix, i.e., that A_i is positive. Let A_i lie in rows k and $k+1$ and columns k and $k+1$ of C. Let the only positive entry of E_i be in row k of C, and that of E_{i+1} in column k of C. It follows that the only nonzero entries in row $k+1$ of C are $c_{k+1,k}$ and $c_{k+1,k+1}$. We assert that $B = (b_{ij}) = C - c_{kk}E_{kk}$ is totally indecomposable. If B were partly decomposable, then it would contain an $s \times (n-s)$ zero submatrix. Note that every row of B has at least two positive entries, and therefore $s \geqslant 2$. Since C is totally indecomposable and differs from B only in the (k, k) position, the zero submatrix must contain b_{kk}. But $b_{k,k+1} = c_{k,k+1}$ and $b_{k+1,k} = c_{k+1,k}$ are both positive, so that the $s \times (n-s)$ zero submatrix must be contained in $B(k+1|k+1)$. However, this implies that the submatrix M obtained from B by deleting columns k and $k+1$ would contain an $(s+1) \times (n-s-1)$ zero submatrix, since all the entries in row $k+1$ of M are zero. We have arrived at a contradiction because M is a submatrix of A, and A is totally indecomposable. Thus we have proved that, if A_i is a 2×2 positive matrix, then $PAQ - c_{kk}E_{kk}$ is totally indecomposable. This, however, contradicts the fact that A is nearly decomposable.

An n-square totally indecomposable matrix, $n \geqslant 2$, must have at least two positive entries in each row and each column. In particular, a nearly decomposable matrix must have this property. Hence it must have at least $2n$ positive entries. In Theorem 2 we show that a nearly decomposable matrix, unlike a general totally indecomposable matrix, can only have relatively few positive entries. It is convenient to state the result in terms of (0, 1) matrices. Let $\sigma(X)$ denote the sum of the entries in the matrix X, and $\pi(X)$ the number of nonzero elements in X. Of course, if X is a (0, 1) matrix, then $\sigma(X) = \pi(X)$.

THEOREM 2. *Let A be an n-square nearly decomposable* (0, 1) *matrix, $n > 2$. Then*

$$\sigma(A) \leqslant 3(n-1). \tag{2}$$

Equality holds in (2) *if and only if there exist permutation matrices P and Q such that*

$$PAQ = \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & 0 & \cdot & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdot & 0 & 0 \\ 1 & 0 & 0 & 1 & \cdot & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 1 & 0 & 0 & 0 & \cdot & 1 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{3}$$

Proof. By Lemma 2, there exist permutation matrices P_1 and Q_1 such that P_1AQ_1 is of the form (1), where A_i is n_i-square, $i = 1, \ldots, r$. We know from Theorem 1 that $n_i \neq 2$, $i = 1, \ldots, r$. Suppose that $n_{i_t} \geqslant 3$ for $t = 1, \ldots, k$, and $n_i = 1$ otherwise. Use induction on n. If $n = 3$, the matrix A must have exactly two ones in each row and in each column, and then $\sigma(A) = 3 \times 2 = 3(n - 1)$. Assume now that $n > 3$ and that the theorem holds for all nearly decomposable matrices of orders greater than 2 and less than n. Let $K = \{i_1, \ldots, i_k\}$. Then

$$\begin{aligned} \sigma(A) &= \sigma(P_1AQ_1) \\ &= \sum_{i=1}^{r} \sigma(A_i) + \sum_{i=1}^{r} \sigma(E_i) \\ &= \sum_{i \in K} \sigma(A_i) + \sum_{i \notin K} \sigma(A_i) + \sum_{i=1}^{r} \sigma(E_i) \\ &\leqslant \sum_{i \in K} (3n_i - 3) + (r - k) + r \\ &= 3\left(\sum_{i \in K} n_i\right) - 3k + 2r - k \\ &= 3[n - (r - k)] - 4k + 2r \\ &= 3n - k - r \\ &\leqslant 3(n - 1), \end{aligned} \tag{4}$$

since $k \geqslant 1$ and $r \geqslant 2$. Equality holds in (2) only if $k = 1$, $r = 2$, and (4) is an equality. Again we use induction on n. The cases $n = 3$ and 4 can be easily proved directly. If $n > 4$ and $k = 1$, $r = 2$, then by Lemma 2 there exist permutation matrices P_1 and Q_1 such that

$$P_1AQ_1 = \left[\begin{array}{c|c} A_1 & E_1 \\ \hline E_2 & 1 \end{array}\right],$$

where A_1 is a nearly decomposable $(n-1)$-square matrix, and each of E_1, E_2 contains exactly one nonzero entry. Now, $\sigma(A_1) = \sigma(A) - 3 = 3((n-1)-1)$. Therefore, by the induction hypothesis, there exist permutation matrices P and Q such that

$$PAQ = \left[\begin{array}{c|c} B & F_1 \\ \hline F_2 & 1 \end{array}\right],$$

where B is of the form (3), and each of F_1, F_2 contains exactly one 1. It remains to prove that the only positive entry in F_1 is in the first row and the only positive entry in F_2 is in the first column.

Let the positive entry of F_1 be in the pth row, and the positive entry of F_2 in the qth column. Suppose first that $p \neq q$, $p \neq 1$, and consider the matrix $G = PAQ - E_{p1}$. Since A is nearly decomposable, the matrix G must be partly decomposable, i.e., it must contain an $s \times (n-s)$ zero submatrix. Since PAQ and G differ only in the $(p, 1)$ position, the zero submatrix of G must include the zero in the $(p, 1)$ position. Now, G has at most three zeros in the first column, and thus the zero submatrix must lie in rows $1, p$, and n. But then it cannot be $s \times (n-s)$. For, if it contains the zero in the $(1, 1)$ position, it cannot be larger than 3×1. If the zero submatrix lies in rows p and n only, it can be at most $1 \times (n-2)$ or $2 \times (n-3)$, since $p \neq q$. Thus, if $p \neq q$ and $p \neq 1$, the matrix G could not be partly decomposable, and hence A could not be nearly decomposable. The same conclusion follows if $p \neq q$ and $q \neq 1$. Now suppose that $p = q$, $p \neq 1$, and consider the matrix $K = PAQ - E_{pp}$. As in the preceding case, K must contain an $s \times (n-s)$ zero submatrix Z, one of whose entries must be the zero in the (p, p) position. Now the first and last entries in the pth row and the pth column are positive, and therefore Z is contained in $K(1, n|1, n)$. (If M is any $m \times n$ matrix and $i_1 < \cdots < i_s \leqslant m$, $j_1 < \cdots < j_t \leqslant n$ are positive integers, then $M(i_1, \ldots, i_s|j_1, \ldots, j_t)$ denotes the submatrix obtained from M by deleting rows $i_1, \ldots, i_s$ and columns $j_1, \ldots, j_t$.) Now consider the submatrix $K(1, p, n|1)$. All but one of the rows of Z are contained in its first $n-2$ columns; also all the entries in the last column of $K(1, p, n|1)$ are zero. Hence $K(1, p, n|1)$ contains an $(s-1) \times (n-s+1)$ zero submatrix. But $K(1, p, n|1) =$

$(PAQ)(1, p, n|1)$ and therefore we are led to the conclusion that $p = q$, $p \neq 1$ implies that A contains an $(s - 1) \times (n - s + 1)$ zero submatrix; this is impossible since A is totally indecomposable. Thus we have shown that the only alternative which is consistent with A being nearly decomposable is $p = q = 1$. But then PAQ is of the form (3). The sufficiency of the condition for equality in the statement of the theorem is obvious.

COROLLARY. *Let A be an n-square nearly decomposable nonnegative matrix, $n > 2$. Then*

$$\pi(A) \leqslant 3(n-1). \tag{5}$$

Equality holds in (5) *if and only if there exist permutation matrices P and Q such that PAQ has the same zero pattern as* (3).

The matrix (3) has some quite remarkable properties. It is the incidence matrix of a configuration occurring in the case of equality of the famed de Bruijn-Erdös theorem [1]. Also, it is easy to see that it belongs to the select small class of matrices whose permanent and absolute value of the determinant are equal. In fact, if A is an n-square matrix of the form (3), $n \geqslant 3$, then

$$\mathrm{per}(A) = |\det(A)| = n - 1.$$

Thus, for this remarkable type of matrix, equality holds in Lemma 3(ii) as well!

REFERENCES

1 N. G. de Bruijn and P. Erdös, *Indag. Math.* **10**(1948), 421–423.
2 M. Marcus and H. Minc, *Amer. Math. Monthly* **72**(1965), 577–591.
3 H. Minc, *Proc. Amer. Math. Soc.* **22**(1969), 117–123.
4 R. Sinkhorn, *Proc. Amer. Math. Soc.* **21**(1969), 197–201.
5 R. Sinkhorn and P. Knopp, *Trans. Amer. Math. Soc.* **136**(1969), 67–75.

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